Hohmann and beyond

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Abstract

This are just some preliminary notes for an introduction to Astrodynamics. They refer often to Kerbal Space Program.

Theorem 1. *(Hohmann) Given two circular orbits with the same centre and lying in the same plane (concentric and complanar) of radii r_1 and $r_f > r_1$ respectively, the two-impulse transfer manoeuvre from the first to the second orbit that minimises the total velocity increment Δv_{TOT} is the transfer along the elliptical orbit of periapsis r_1 and apoapsis r_f, the same orbital plane as the two given orbits and the same direction as the starting orbit.*

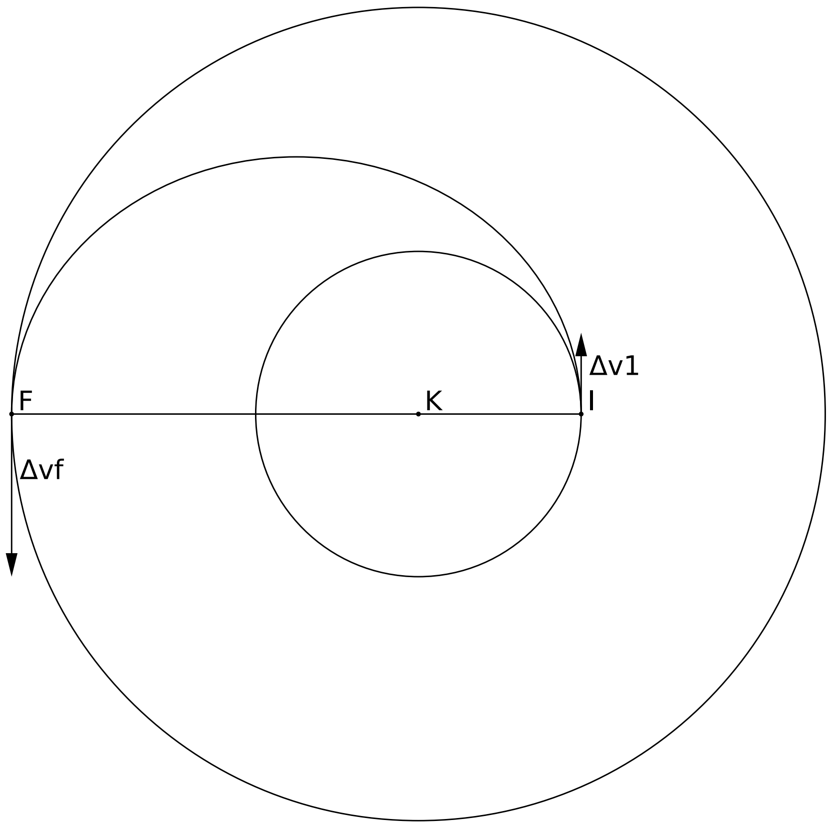

Figure 1. *Hohmann transfer*

Remark 2. Possible situations: launch from the surface of a planet, transfer between two circular orbits around Kerbin.

Proof. Set $\mu = M \cdot G$. We have that the velocity of the first circular orbit is $\overrightarrow{v_{c_1}}$, with $v_{c_1} = \sqrt{\frac{\mu}{r_1}}$, while the final velocity is the one of the circular orbit of radius r_f, i.e. $\overrightarrow{v_{c_f}}$, with $v_{c_f} = \sqrt{\frac{\mu}{r_f}}$. Other variables of interest: $\vec{v}_1$, velocity of the first impulse, $\vec{v_f}$, one of the final impulse, φ is the initial elevation, i.e. the angle between $\vec{v}_1$ and $\overrightarrow{v_{c_1}}$, while ψ is the final one, the angle between $\vec{v_f}$ and $\overrightarrow{v_{c_f}}$.

We are aiming to minimise

$$\Delta v_{\text{TOT}} = |\vec{v}_1 - \vec{v_{c_1}}| + |\vec{v_{c_f}} - \vec{v_f}| = \Delta v_1 + \Delta v_f$$

We can simplify the problem at hand by using the preserved quantities:

$$E = \frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{v_f^2}{2} - \frac{\mu}{r_f} \quad \Longrightarrow \quad v_f^2 = v_1^2 - \frac{2\,\mu\,(r_f - r_1)}{r_1 \cdot r_f}$$

(energy, from vis-viva equation) and (impulse / angular momentum)

$$c = r_1\,v_1 \cos\varphi = r_f\,v_f \cos\psi \quad \Longrightarrow \quad \cos\psi = \frac{r_1}{r_f}\frac{v_1}{v_2}\cos\varphi.$$

Now we can take advantage of those to express both Δv_1 and Δv_f in terms of v_1:

$$\Delta v_1 = |\vec{v}_1 - \vec{v_{c_1}}| = \sqrt{v_1^2 + v_{c_1}^2 - 2\,v_1\,v_{c_1}\cos\varphi}$$

$$\Delta v_f = |\vec{v_f} - \vec{v_{c_f}}| = \sqrt{v_f^2 + v_{c_f}^2 - 2\,v_f v_{c_f}\cos\psi} = \sqrt{v_{c_f}^2 + v_1^2 - \frac{2\,\mu\,(r_f - r_1)}{r_f r_1} - 2\,v_{c_f}\,v_1\frac{r_1}{r_f}\cos\varphi}$$

We can rescale everything in terms of the speed in the internal orbit, $v_{c_1} = \sqrt{\mu/r_1}$ by setting $x = v_1/v_{c_1}$. We get then

$$\frac{\Delta v_{\text{TOT}}}{v_{c_1}} = \sqrt{x^2 - 2\,x\cos\varphi + 1} + \sqrt{x^2 - 2\left(\frac{r_1}{r_f}\right)^{3/2}\cos\varphi - \frac{r_1 - 2\,(r_f - r_1)}{r_f}} = D(x, \varphi)$$

and we want to minimise $D(x,\ \varphi)$, with the additional constraint of the orbit having apoapsis greater or equal than r_f (because we want to reach that orbit). If we consider the partial derivative with respect to φ we have:

$$\frac{\partial D}{\partial\varphi} = \frac{v_1\,v_{c_1}}{\Delta v_1}\sin\varphi + \frac{v_1\,v_{c_f}\,{}^{r_1}/_{r_f}}{\Delta v_f}\sin\varphi$$

we have that the minimum is attained in $\varphi = 0$, independently of the speed, hence of x. We can then consider only the case $D(x) = D(x, 0)$ and differentiate in x:

$$\frac{dD}{dx} = v_{c_1}\frac{x-1}{\Delta v_1} + v_{c_1}\frac{x - ({}^{r_1}/_{r_f})^{3/2}}{\Delta v_f}.$$

We can observe that $x > 1$, otherwise we'd have apoapsis at r_1, and then it follows that the derivative is always positive and the solution to our problem is the minimum value of x (i.e. the minimum value of v_1) such that we can actually reach the final orbit. For obvious geometrical reasons it corresponds to the value v_1 such that the ellipse is tangent in the apoapsis too, i.e.

$$\frac{v_1^2}{2} - \frac{\mu}{r_1} = -\frac{\mu}{r_1 + r_f} \qquad\qquad v_1^2 = v_{c_1}^2\left(2 - \frac{2}{1 + {}^{r_f}/_{r_1}}\right).$$

This covers the case where the transfer orbit lies on the same plane of the initial and final orbit. Leaving the plane is in this case not such a great idea, as we'd just add more terms to Δv_{TOT}. □

Remark 3. The time needed for the Hohmann transfer is the time required for travelling half the ellipse, that is

$$T_H = \pi \sqrt{\frac{\left(\frac{r_1 + r_f}{2}\right)^3}{\mu}}.$$

We can also express $\frac{\Delta v_{\mathrm{TOT}}}{v_{c_1}}$ in terms of $\frac{r_f}{r_1}$, which will turn out to be useful to compare Hohmann transfer to other manoeuvres. We have

$$\begin{aligned}\frac{\Delta v_{\mathrm{TOT}}}{v_{c_1}} &= \frac{\Delta v_1}{v_{c_1}} + \frac{\Delta v_f}{v_{c_f}} = \frac{v_1 - v_{c_1}}{v_{c_1}} + \frac{v_{c_f} - v_f}{v_{c_1}} \\ &= \sqrt{\frac{2 \cdot {}^{r_f}/_{r_1}}{1 + {}^{r_f}/_{r_1}}} - 1 + \sqrt{\frac{1}{{}^{r_f}/_{r_1}}} - \sqrt{\frac{2}{{}^{r_f}/_{r_1}\,(1 + {}^{r_f}/_{r_1})}}.\end{aligned}$$

An easy computation provides us with the interesting insight that the maximum of this ratio is attained for ${}^{r_f}/_{r_1} = 15.58$, in which case $\Delta v_{\mathrm{TOT}} = 0.536\, v_{c_1}$.

Remark 4. If the orbits are elliptical due to the Oberth effect it is usually optimal to leave the internal orbit at the periapsis. We still need to check both cases, as the configuration of the two orbits might cause it to be better to leave the internal orbit at the apoapsis.

Rocket equation and specific impulse Why do we want to reduce Δv anyway? Let's consider a rocket flying with velocity $\vec{v}$ and subject to some external forces of sum $\vec{F}$ (e.g. gravitational, atmospheric friction...). Let $\vec{v_e}$[1] be the ejection velocity of the propellant (with respect to the rocket) in a (infinitesimal) time $\mathrm{d}t$. We want to get the module Δv of the corresponding velocity variation of the rocket. We set the mass of the rocket (including engine mass and fuel mass and payload mass) to m. In a time dt the rocket is subject to the following variation of momentum:

$$[(m - \mathrm{d}m)\,(\vec{v} + \mathrm{d}\vec{v}) + \mathrm{d}m\,(\vec{v} - \vec{v_e})] - m\,\vec{v} = \vec{F}\,\mathrm{d}t$$

where in the first term we have the difference between the momentum at time $t_0 + \mathrm{d}t$ and the momentum at time t_0. Since external forces are not impulsive, we have that $\vec{F}\,\mathrm{d}t \to 0$. If we consider a first order approximation and forget about the second order term $\mathrm{d}m\,\mathrm{d}\vec{v}$, we get

$$m\,\mathrm{d}\vec{v} = \vec{v_e}\,\mathrm{d}m,$$

which projected along $\vec{v}$ gives

$$\mathrm{d}v = -v_e \frac{\mathrm{d}m}{m},$$

hence

$$\Delta v = v_e \log\left(\frac{m_0}{m_f}\right)$$

1. Maybe a different notation is better, as for masses we have m_e denoting the mass of the engine... So maybe v_f (m_p is the mass of the payload)

with m_0 the mass of the rocket at time 0, and m_f the mass at the end of the manoeuvre. The fuel used for the Δv variation is then $m_0 - m_f$. It makes sense to introduce the following quantity, called specific impulse:

$$I_{\text{sp}} = \frac{v_e}{g}$$

with g being the gravitational acceleration at sea level (9.81 on Kerbin or Earth). The specific impulse (which has seconds as its unit, at least in the most common formulation, given above), is a measure of the efficiency of the engine used (namely it depends on the engine and the fuel used). As it is clear from the previous equation for the Δv, the higher the specific impulse is, the better.

Engine	I_{sp} **(vac)**	**mass**	**thrust**	**TWR**
Rocketdyne F1	263 s	8400 kg	6.77 MN	82.27
R25	452.3 s	3526 kg	2.28 MN	65.91
Merlin 1D	340 s	490 kg	0.8 MN	150
RD 171 M	337 s	9300 kg	7.9 MN	86.9
RD 107 A	310 s	1190 kg	1 MN	89.9
PPS 1350 1.5 kW	1650s	5.3 kg	0.088 N	0.0017

Table 1. Earth engines

Engine	I_{sp} **(vac)**	**mass**	**thrust**	**TWR**
Rockomax Mainsail	310 s	6000 kg	1.5 MN	25.48
LV-T45	320 s	1500 kg	0.2 MN	13.59
LV-N*	800 s	3000 kg	0.06 MN	2.04
PB-ION	4200 s	250 kg	2 kN	0.816

Table 2. Kerbal engines

Might be worth noting that usually (rocket) engines have different values for the specific impulse depending on being or not in the atmosphere (compared to be in the vacuum). Also worth stressing is that this rocket equation (or Tsiolkovskij equation) holds for engines that (approximately) fire in a single impulse, and does not hold for engines such as electrical ones (very low thrust, even if they have a huge Isp).

Moreover it is not necessarily a good measure of comparison for engines, not just for problems as the one above (thrust vs isp, instant vs long time), but also because an engine with a very high Isp might have a huge mass, thus using a lot of its power to move itself, so it is better to compare the work provided by each engine. Reminder: thrust is given by

$$\vec{T} = \vec{v} \cdot \frac{\mathrm{d}m}{\mathrm{d}t}$$

and its unit is Newtons, while thrust to weight ratio (TWR) by

$$\text{TWR} = \frac{T}{m \cdot g}$$

and is a pure number.

Bi-elliptic transfer and comparison with Hohmann. The Hohmann theorem above tells us that the Hohmann transfer is optimal among those with 2 impulses, but what if we relax this hypothesis and consider the case with 3 impulses? We get the bi-elliptic transfer.

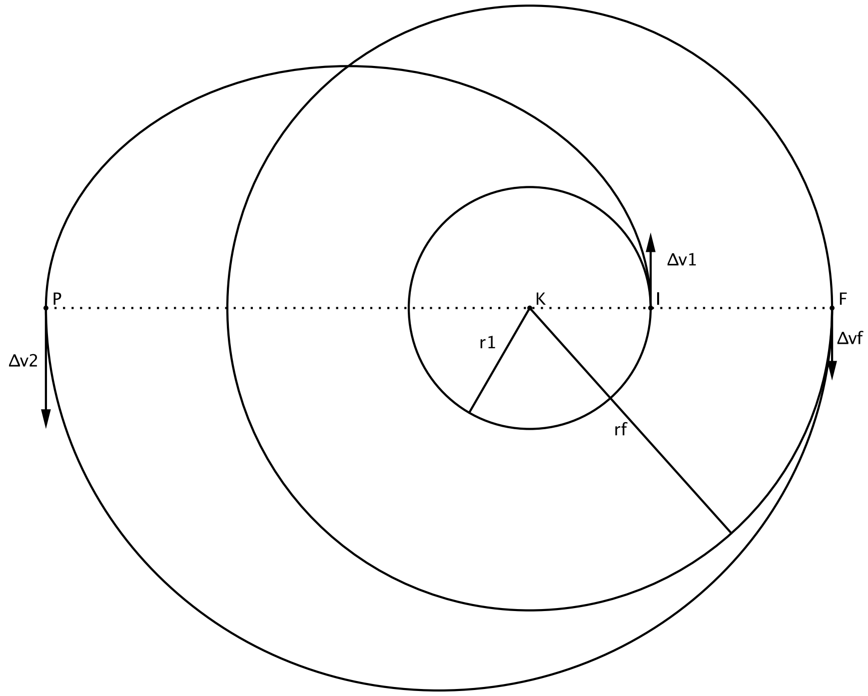

Figure 2. Bielliptic transfer

So we start once again in I, on the circular orbit of radius r_1, we have a first burn Δv_1 along the same direction as v_{c_1} (prograde) and we start the first elliptical transfer orbit, with semi-major axis $a_1 = \frac{r_1 + r_2}{2}$, where we stay until we reach the apoapsis P (lying at a distance $r_2 > r_f$ from our gravity centre). In this point we burn prograde again (Δv_2), raising the periapsis to height r_f and starting a second elliptical transfer orbit of semi-major axis $a_2 = \frac{r_2 + r_f}{2}$. When we reach the periapsis of this orbit (at F), it is time for the third and last burn Δv_f, which is retrograde and circularises the orbit.

Using the same ideas as for the Hohmann Theorem, we can compute:

$$\frac{v_1^2(I)}{2} - \frac{\mu}{r_1} = -\frac{\mu}{r_1 + r_2}, \quad \frac{v_1^2(P)}{2} - \frac{\mu}{r_2} = -\frac{\mu}{r_1 + r_2}, \quad \frac{v_2^2(P)}{2} - \frac{\mu}{r_2} = -\frac{\mu}{r_2 + r_f}, \quad \frac{v_2^2(F)}{2} - \frac{\mu}{r_f} = -\frac{\mu}{r_2 + r_f}.$$

And now, since

$$\Delta v_1 = v_1(I) - v_{c_1}, \qquad \Delta v_2 = v_2(P) - v_1(P), \qquad \Delta v_f = v_2(F) - v_{c_f},$$

we can compute the three Δv (or better, their ratio with v_{c_1}):

$$\frac{\Delta v_1}{v_{c_1}} = \sqrt{\frac{2 \cdot {}^{r_2}/_{r_1}}{1 + {}^{r_2}/_{r_1}}} - 1, \qquad \frac{\Delta v_2}{v_{c_1}} = \sqrt{\frac{2 \cdot {}^{r_f}/_{r_1}}{{}^{r_2}/_{r_1}\,({}^{r_f}/_{r_1} + {}^{r_2}/_{r_1})}} - \sqrt{\frac{2}{{}^{r_2}/_{r_1}\,(1 + {}^{r_2}/_{r_1})}},$$

and

$$\frac{\Delta v_f}{v_{c_1}} = \sqrt{\frac{2 \cdot {}^{r_2}/_{r_1}}{{}^{r_f}/_{r_1}\,({}^{r_f}/_{r_1} + {}^{r_2}/_{r_1})}} - \sqrt{\frac{1}{{}^{r_f}/_{r_1}}}.$$

Now we can set $x = {}^{r_f}/_{r_1}$ and $y = {}^{r_2}/_{r_1}$ and we can write ${}^{\Delta v_B}/_{v_{c_1}}$ in term of these as

$$\frac{\Delta v_B}{v_{c_1}} = \sqrt{\frac{2 \cdot y}{1 + y}} - 1 + \sqrt{\frac{2 \cdot x}{y\,(x + y)}} - \sqrt{\frac{2}{y\,(1 + y)}} + \sqrt{\frac{2 \cdot y}{x\,(x + y)}} - \sqrt{\frac{1}{x}}.$$

We are interested in when this manoeuvre is better than Hohmann, i.e. when its fuel consumption is less, i.e. when the Δv required is less, so we study where $\Delta v_B - \Delta v_H < 0$, or equivalently when ${}^{\Delta v_B}/_{v_{c_1}} - {}^{\Delta v_H}/_{v_{c_1}} < 0$. The answer is provided by this inequality (holding for $y > x$, i.e. when $r_2 > r_f$):

$$\sqrt{\frac{2\cdot y}{1+y}} - 1 + \sqrt{\frac{2\cdot x}{y\,(x+y)}} - \sqrt{\frac{2}{y\,(1+y)}} + \sqrt{\frac{2\cdot y}{x\,(x+y)}} - \sqrt{\frac{1}{x}} - \sqrt{\frac{2\cdot x}{1+x}} + 1 - \sqrt{\frac{1}{x}} + \sqrt{\frac{2}{x\,(1+x)}} < 0$$

and is represented in the following picture:

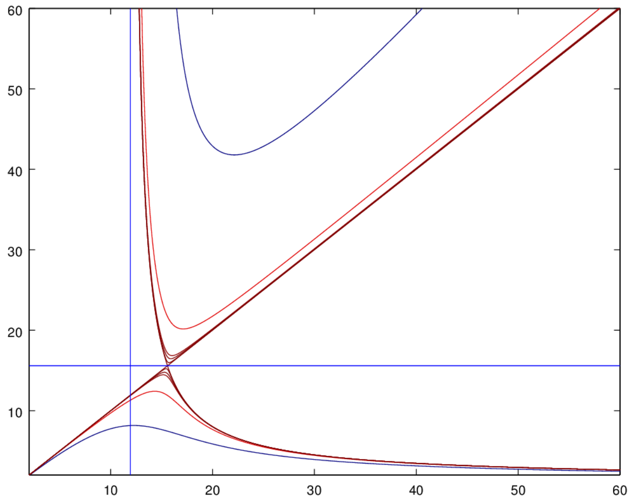

Figure 3. Comparison between Hohmann and bielliptic

So, when ${}^{r_f}/_{r_1} > 15.581$ any choice of $r_2 > r_f$ provides some gain, while for ${}^{r_f}/_{r_1} < 11.94$ Hohmann transfer is always better. In between if we take r_2 big enough we can gain with a bielliptic transfer. The best case is for $r_2 = \infty$, for which we'd have $\Delta v_B = (\sqrt{2} - 1)\left(\sqrt{\frac{\mu}{r_1}} + \sqrt{\frac{\mu}{r_f}}\right)$. This case shows us the problem that we might have with a bielliptic transfer: we have to travel two half ellipses, so the total time is

$$T_B = \pi\sqrt{\frac{\left(\frac{r_1 + r_2}{2}\right)^3}{\mu}} + \pi\sqrt{\frac{\left(\frac{r_2 + r_f}{2}\right)^3}{\mu}}.$$

Anyway, if we are not in a hurry, for which kind of missions can we use such manoeuvre to save fuel, i.e. when does it happen that the ratio of the radii is such that it allows a bielliptic transfer? In KSP we can use it to travel from LKO (or MKO) to an higher orbit or to the Mun (or Minmus, but we have a problem...). On the other hand the bielliptic manoeuvre is not suitable for transfers between planets, as the Kerbol System is quite small: even if we were to send a prob from Kerbin to Eeloo when it is at his furthest it would be a ratio of 8.69 between the radii, so that would not be fuel sparing.

On the other hand in our good ol' Solar System we have not only the transfers between orbits around the Earth, but also from LEO (or MEO) to the Moon, and from Earth to Uranus (${}^{r_f}/_{r_1} \approx 19$), to Neptune (${}^{r_f}/_{r_1} \approx 30$) or to Pluto (${}^{r_f}/_{r_1} \approx 39$) and beyond...

Remark 5. It might also be worth noting that the highest possible saving of Δv with a bielliptic transfer compared to a Hohmann transfer is 8%, so nothing spectacular, but still not quickly discarded, considering that a big part of the mass of a spacecraft consists of fuel.

Change of plane. Before, when I mentioned Minmus, I stated that we have a problem with that, and the problem is that Minmus' orbit lies on a different plane than an equatorial orbit around Kerbin. So moving to Minmus requires a change of plane. Let's see how we do it! We start (and finish) with a circular orbit of radius r, so their speed is

$$v_{c_1} = v_{c_2} = v_c = \sqrt{\frac{\mu}{r}},$$

and if we are changing the plane by an angle ϑ we have

$$\Delta v = 2\, v_c \sin {}^{\vartheta}\!/_{2}, \qquad \frac{\Delta v}{v_c} = 2 \sin {}^{\vartheta}\!/_{2}.$$

This means that to change the plane of an angle $\vartheta = 60°$, we need $\Delta v = v_c$, while to change it of an angle of $180°$ (i.e. to reverse orbit) we need $\Delta v = 2\, v_c$ (very reasonable).

Note that we fire in a direction which is orthogonal to both the tangent to the orbit (we want the orbit to stay the same, radius-wise) and the vector to the center of mass. This direction is called the normal direction. Also note that the point where we execute the burn is a point that lies on both the original orbit and the new one, i.e. is in the intersection of the two planes. One must not think of the point where we burn as rising in space.

Remark 6. The further away we are from the gravity centre, the cheaper the manoeuvre is.

This remark tells us something: maybe we can gain some fuel if we send the probe far enough so that the Δv required for the change of orbit, change plane and come back is less than the fuel necessary to change plane staying in the same orbit. So we aim for a 3-burns manoeuvre.

The first prograde burn of Δv_1 takes us on an orbit, on the same plane, of apoapsis $r_f > r_1$, then we have a manoeuvre that just changes the plane (while we are at the apoapsis), by a burn of $\Delta v_2 = 2\, v_2 \sin {}^{\vartheta}\!/_{2}$, and finally we have a retrograde burn at the periapsis to lower the apoapsis back to r_1 and circularise the orbit once again. The Δv for this last burn is exactly the same needed for Δv_1, so the total is

$$\Delta v_{\text{TOT}} = 2\, \Delta v_1 + \Delta v_2.$$

Now we use the same change of variables as before, i.e. $x = {}^{r_f}\!/_{r_1}$ and we consider

$$\frac{v_2^2}{2} - \frac{\mu}{r_f} = -\frac{\mu}{r_1 + r_f} \qquad \Longrightarrow \qquad v_2 = v_c \sqrt{\frac{2}{x\,(1+x)}},$$

whence

$$\frac{\Delta v_2}{v_c} = 2 \sqrt{\frac{2}{x\,(1+x)}} \sin \frac{\vartheta}{2}.$$

At the same time we have

$$\frac{2\, \Delta v_1}{v_c} = 2 \left(\sqrt{\frac{2\,x}{1+x}} - 1 \right)$$

and we can proceed to minimise ${}^{\Delta v_{\text{TOT}}}\!/_{v_c}$ with respect to x, for fixed ϑ. We get (computations!!! Insert in future, maybe) the following answer:

$$x_{\text{ott}} = \frac{\sin {}^{\vartheta}\!/_{2}}{1 - 2 \sin {}^{\vartheta}\!/_{2}},$$

which holds for $x_{\text{ott}} \geqslant 1$. We can check how, for fixed x, the function that we are minimising is is behaving wrt ϑ. We go and see when it happens that $x_{\text{ott}} = 1$, which is for $\sin \vartheta/2 = 1/3$, i.e. for $\vartheta \approx 38.94°$. So for $0 < \vartheta < 38.94°$ we have that a single burn manoeuvre is more efficient, for $38.94° \leqslant \vartheta \leqslant 60°$ we can have a more efficient 3-burns manoeuvre by choosing $x = x_{\text{ott}}$. Actually, any $1 < x \leqslant x_{\text{ott}}$ provides some improvement wrt single burn. When we reach 60° we hit $x_{\text{ott}} = \infty$, so any choice of $x > 1$ gives some benefit, and the bigger the x the better. (The reason is that the derivative is always negative, but increasing to 0)

Again for practical reasons we will choose some x that gives us enough benefits while it doesn't take too long (the time taken is the time for an elliptic orbit of semi-major axis $\frac{r_1 + r_f}{2}$).

Remark 7. There is an obvious generalisation to this manoeuvre, that is to split the angle in 3 contributes $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$ and include each of them in one of the three burns. This way we can perform even better.

Hohmann transfer with change of plane. If we need to change the plane we can split the contributions in two, one of each burn. This is exactly the idea of the previous remark. To choose the optimal α to be taken care of in the first burn, we need to solve numerically a minimisation problem.

Flybys. We can use gravity to save fuel. We get the so-called slingshot or gravity assist. We can use that to accelerate or decelerate, or just to modify our trajectory. For example the Cassini spacecraft was able to reach Saturn with 2km/s Δv instead of the 15.7 km/s required by the Hohmann transfer thanks to the 2 flybys with Venus, one with the Earth and one with Jupiter. From a time perspective it needed 6.7 years instead of the 6 required by Hohmann. Also Ulysses (change of plane, out of the ecliptic). Problem: you can't force the planets to be where you want them! There are situations where this is not a problem: for example in the Mun landing tutorial you can use a flyby with the Mun (clockwise) to slow your craft down, saving fuel for a soft landing (in contraposition with hard landing, jargon for "crashing").

We already mentioned in passing the Oberth effect: it states that a prograde manoeuvre is the most efficient if performed at the periapsis. The Oberth effect is easy to prove formally, but is trickier to understand from an intuitive point of view. We have that when we perform the burn the whole craft + fuel system preserves the kinetic energy (because the potential energy is the same in the point), so we have

$$\frac{1}{2}(M + m)\, v^2 = \frac{1}{2} M\, (v + \Delta v)^2 + \frac{1}{2} m\, (v - v_e)^2.$$

On the other hand, if we consider only the spacecraft we have

$$\Delta E = \frac{1}{2} M\, (v + \Delta v)^2 - \frac{1}{2} M v^2 = \frac{1}{2} M\, \Delta v^2 + M v\, \Delta v$$

and so we see that the higher v is, the more efficient the burn, as we gain more kinetic energy than just $1/2\, M\, \Delta v^2$. We could say that we are "stealing" it from the fuel that we are leaving behind.

Of course one can combine the two techniques presented in a powered flyby, the only problem being, in real life, the precision of such manoeuvre and the timing issues due to communications lags (try playing with RemoteTech...).

What else? Something we can't use in KSP, due to the SOI and 2 bodies approximation (hence no Lagrangian points and so on... Low energy transfers, ballistic capture and Interplanetary transport network. Just a passing remark: on an Earth-Moon LET the fuel savings can be up to 25%.

Free return trajectory. Or how to bring tourists around the Mun, get some science and land back on Kerbin with a single burn from Kerbin orbit! Philipp would complain about my drawing, here, so I will just mention it.

books

code