I

$$f(x) := \sum_{n=0}^{\infty} a_n x^n$$

Generating Functions

$a_n = \alpha n + \beta$ für $n \geq 0$

$$a_n = \alpha n + \beta \quad / \cdot x^n + \text{sum}$$

$$\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{f(x)} = \sum_{n=0}^{\infty} (\alpha n + \beta) x^n$$

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n \quad \text{mit} \quad c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$$

RS:

$$\ldots = \underbrace{\alpha \sum_{n=0}^{\infty} n x^n}_{\alpha x \frac{d}{dx} \sum_{n=0}^{\infty} x^n} + \underbrace{\beta \sum_{n=0}^{\infty} x^n}_{= \frac{\beta}{1-x}}$$

$$\underbrace{\alpha x \frac{d}{dx} \sum_{n=0}^{\infty} x^n}_{\alpha x \frac{d}{dx}\left(\frac{1}{1-x}\right)}$$

$$\underbrace{\alpha x \frac{d}{dx}\left(\frac{1}{1-x}\right)}_{\ldots}$$

$$\Rightarrow f(x) = \frac{\alpha x}{(1-x)^2} + \frac{\beta}{1-x} = \frac{x(\alpha - \beta) + \beta}{(1-x)^2}$$

Fibonacci: 0, 1, 1, 2, 3, 5, 8, ...

$$F_{n+1} = F_n + F_{n-1}, \text{ für } n \geq 1, \ F_0 = 0, F_1 = 1$$

LS:

$$\sum_{n=1}^{\infty} F_{n+1} x^n = F_2 x + F_3 x^2 + F_4 x^3 + \dots$$

$$\boxed{\text{define: } F(x) = \sum_{n=1}^{\infty} F_n x^n}$$

$$= \frac{F_1 x + F_2 x^2 + \dots - F_1 x}{x} = \frac{F(x) - x}{x}$$

RS:

$$\underbrace{\sum_{n=1}^{\infty} F_n x^n}_{F(x)} + \underbrace{\sum_{n=1}^{\infty} F_{n-1} x^n}_{F_0 x + F_1 x^2 + \dots = F(x) \cdot x}$$

$$\Longrightarrow \frac{F(x) - x}{x} = F(x) + x F(x)$$

$$\Longleftrightarrow F(x) = \frac{x}{1 - x - x^2}$$

$$f \xleftrightarrow{ops} \{a_n\}_0^\infty \Longrightarrow f = \sum_{n=0}^{\infty} a_n x^n$$

$$f \xleftrightarrow{ops} \{a_{n+1}\}_0^\infty \Longrightarrow \sum_{n=0}^{\infty} a_{n+1} x^n = \frac{f(x) - f(0)}{x}$$

<u>1. Regel</u>: $h > 0$: $\{a_{n+1}\} \Longrightarrow \sum_{n=0}^{\infty} a_{n+h} x^n = \frac{f(x) - a_0 - \ldots - a_{h-1} x^{h-1}}{x^h}$.

<u>2. Regel</u>: P Polynom : $P(xD) \xleftrightarrow{ops} \{P(n) a_n\}_0^\infty$

II $\quad f = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$

$$f \xleftrightarrow{egf} \{a_n\}_0^\infty$$

$$\{a_{n+1}\}_0^\infty$$

$$\sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} = \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{n a_n x^{n-1}}{n!} = f'$$

<u>1. Regel</u>: für $h > 0$

$$\{a_{n+h}\}_0^\infty \xleftrightarrow{egf.} D^{(h)} f.$$

<u>Fibonacci</u>: $\quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0)$. $F_0 = 0, F_1 = 1$.

$$\Longrightarrow f'' = f' + f \Longrightarrow f = c_1 e^{r_+ x} + c_2 e^{r_- x}$$

wobei $r_\pm = \frac{1 \pm \sqrt{5}}{2}$

$c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$

$$\Longrightarrow f = \frac{e^{r_+ x} - e^{r_- x}}{\sqrt{5}}$$

$$\left[\frac{x^n}{n!}\right] f = \frac{e^{r_+ x} - e^{r_- x}}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(r_+^n - r_-^n\right)$$

$$\left[\frac{x^n}{n!}\right] e^x = 1.$$

Bellsche Zahlen : 1, 1, 2, 5, 15, 52

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$$

$$B_n = \frac{1}{e}\sum \frac{1}{k!(1-kx)}$$

$$f(x) = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)\frac{x^n}{n!}$$

$$= \frac{1}{e}\sum_{k=0}^{\infty}\left(\frac{1}{k!}\underbrace{\sum_{n=0}^{\infty}\frac{x^n k^n}{n!}}_{e^{x\cdot k}}\right) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{(e^x)^k}{k!}$$

$$= \frac{1}{e}e^{e^x} = e^{e^x-1}.$$

Weitere Regeln :

2. Regel : $P(xD) f \overset{egf}{\longleftrightarrow} \{P(n)\, a_n\}$, $n \geq 0$.

$$f, g \quad \left[\frac{x^n}{n!}\right] f\cdot g = \sum_{r,s>0}^{\infty}\frac{a_r b_s}{r!\, s!} x^r x^s \quad , \; s = n-r$$

$$= \sum_{n-r,\, r>0}^{\infty}\frac{a_r b_{n-r}}{r!\,(n-r)!}\underbrace{x^r x^{n-r}}_{x^n}$$

$$\Longrightarrow (fg) = \sum_r \binom{n}{r} a_r b_{n-r}.$$

$$b(n+1) = \sum_k \binom{n}{k} b(k).$$

$b(n)$ = Anzahl der Partitionen der Menge $\{1, \ldots, n\}$

Partitionen von $\{1, \ldots, n+1\}$

Gegeben Partition von $\{1, \ldots, n+1\}$

<u>Frage</u>: Wo liegt das Element $n+1$?

~~Nimm raus → erhalte Partition von $\{1, \ldots, n\}$~~

Nimm diese Menge raus! Diese Menge haben $n-k+1$ Elemente.

→ Erhalte Partition von k vielen Elementen

Welche k-viele Elemente ?

$\binom{n}{k}$ Möglichkeiten.

$$b' = b \cdot e^x \implies b = c \cdot \exp(e^x)$$

$$c = \frac{1}{e} \; ; \; 1 = c \cdot e \implies b = \frac{1}{e} \exp(e^x)$$

$$\implies e^{e^x - 1}$$