Differenzenrechnung

Grundidee: Betrachte Funktionen auf den ganzen Zahlen

$f: \mathbb{Z} \to \mathbb{C}$

$\Delta f(x) = f(x+1) - f(x) = \underset{,}{E}f - f|_x = (E-1)f|_x$

Verschiebeoperator

Eigenschaften

- $\partial(fg) = (\partial f)g + f(\partial g)$
- ~~$\Delta(fg)(x) = (\Delta f)g$~~

  $\Delta(fg)|_x = (\Delta f)g|_x + (\Delta g)|_x f(x+1)$

  $= (\Delta f)g|_x (\Delta g)Ef|_x$

- $\Delta^k f = (E-1)^k f = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} E^j f$

  insb.: $(\Delta^k f)(0) =$

  $= \sum \dots f(j)$

- $\Delta f(x+a) = (\Delta f)(x+a)$
- $\sum_{k=a}^{b-1} \Delta f(k) = f(b) - f(a)$
- Partielle Integration

  $\sum_a^b \Delta(fg) = \sum_a^b \Delta f g + \sum_a^b (Ef)\Delta g$

- $\cancel{\partial x^n} \ne n \cancel{x^{n-1}}$ Wünschenswert Regel wie $\partial x^n = n x^{n-1}$

aber Problem kein limes!

Kombinatorischer Einschub

Erinnerung: Sterling-Zahl 2. Art:

$\left\{ {n \atop k} \right\}$ = Anzahl der k-elementigen Partitionen von $\{1, \dots, n\}$

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\}$$

$$\left\{ {0 \atop 0} \right\} = 1 \quad , \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$$

~~Betrachte die~~ Anzahl der surjektiven Abbildungen von $\{1, \dots, n\} \to \{1, \dots, k\}$

$$\left\{ {n \atop k} \right\} k!$$

Betrachte $\{ f^{-1}[X] : x \in \{1, \dots, k\}\}$ ist Partition –

Sterling Zahl: ~~2~~1. Art

$\left| \left[{n \atop k} \right] \right|$ = Anzahl der Unterteilungen von $\{1, \dots, n\}$ in k Zykeln.
(→ k Anzahl der Zykeln beliebiger Länge)

Beispiel: Zykel

$$(1, 2, 3) = (2, 3, 1) = (3, 1, 2)$$

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd}$$

$$\left|\begin{bmatrix} n+1 \\ k \end{bmatrix}\right| = \left|\begin{bmatrix} n \\ k-1 \end{bmatrix}\right| + n\left|\begin{bmatrix} n \\ k \end{bmatrix}\right|$$

Randbedingungen

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ n \end{bmatrix}$$

Anzahl der injektiven Abb. von

$\{1, \dots, k\} \to \{1, \dots, n\}$: $\binom{n}{k} k!$

zurück zur Forderung

$$\Delta x^{\underline{n}} \overset{!}{=} n x^{\underline{n-1}}$$

$$x^{\underline{n}} = \prod_{j=0}^{n-1} (x-j) = x(x-1)\cdots(x-n+1) \,, \quad n \geq 1$$

$$x^{\underline{0}} = 1 \qquad x^{\underline{1}} = x \qquad x^{\underline{2}} = x(x-1)$$

$$x^{\underline{x+1}} = 0 \text{, falls } x \in \mathbb{N}$$

$$x^{\underline{n}} = \left(\prod_{j=1}^{-n} (x+j) \right)^{-1} = \frac{1}{(x+1)(x+2)\cdots(x-n)}$$

Kompaktere Schreibweise

$$x^{\underline{n}} = \frac{x!}{(x-n)!}$$

$$\begin{aligned}
\Delta x^{\underline{n}} &= \Delta \prod_{j=0}^{n-1} (x-j) \\
&= \prod_{j=0}^{n-1} (x+1-j) - \prod_{j=0}^{n-1} (x-j) \\
&\cancel{= \left[\prod_{j=1}^{n-2} (x-j) \right] \left((x+n) - x \right) = n x^{\underline{n-1}}} \\
&= (x+1)x(x-1)\cdots(x-n+2) - x(x-1)\cdots \\
&\qquad \cdots (x-n+1)
\end{aligned}$$

$$= (x+1)(\cdots) - (\cdots)(x-n+1)$$

$$= n(\cdots) = n\, x^{\underline{n-1}}$$

Beispiel

$$\partial^k x^n = n(n-1)(n-2)\cdots(n-k)\, x^{n-k}$$

$$= n^{\underline{k}}\, x^{n-k}$$

$$\Delta^k x^{\underline{n}} = n^{\underline{k}}\, x^{\underline{n-k}}$$

„Binomische Formel"

$$(a+b)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} a^{\underline{n-k}}\, b^{\underline{k}}$$

Taylorentwicklung

f beliebig glatte Funktion. Finde Darstellung von f als Potenzreihe

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k = \sum_{k=0}^{\infty} \frac{\partial^k f(0)}{k!} x^k$$

$$\partial^j f|_x = \sum_{k=0}^{\infty} \alpha_k\, k(k-1)x^{k-1}\cdots k(k-j+1)\, x^{k-j}$$

$$\partial^j f|_0 = \alpha_j\, j!$$

$$\Rightarrow \alpha_j = \frac{\partial^j f|_0}{j!}$$

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \quad , x \in \mathbb{Z}$$

$$\frac{(\Delta^j f)(0)}{j!} = \alpha_j$$

$$\cancel{\frac{\Delta_a^j (a+b)^{\underline{n}}}{j!}} (a=0) = n(n-1)\cdots(n-j+1) \cdot \underset{\overset{\shortparallel}{0}}{(a+b)}^{\underline{n-j}}$$

$$= \frac{n^{\underline{j}}}{j!} = \binom{n}{j}$$

Wollen: $$x^{\underline{n}} = \sum_{j=0}^{\infty} \left[{n \atop j}\right] \cdot x^j =$$

$$= \left[{n \atop j}\right] + \sum_{k=1}^{\infty} \left(\left[{n-1 \atop k-1}\right] - (n-1) \left[{n-1 \atop k}\right] \right) x^k$$

$$= \left[{n \atop 0}\right] + x \sum_{k=0}^{\infty} \left[{n-1 \atop k}\right] x^k - (n-1) \left[{n-1 \atop 0}\right] - (n-1) \cdot \sum_{k=0}^{\infty} \left[{n-1 \atop k}\right] x^k$$

$$\cancel{= x x^{\underline{n}} - (n-1) x^{\underline{n-1}}}$$

→ Haben erzeugende Funktion für Sterling Zahlen 1. Art

wollen: $$x^n = \sum_{k=0}^{n} \left\{{n \atop k}\right\} x^{\underline{n}}$$

$$\frac{\Delta^k x^n}{k!} = \frac{1}{k!} (E-1)^k \lambda^n |_0$$

$$= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \underbrace{E^j x^n}_{j^n} |_0 = \left\{{n \atop k}\right\}$$

$$\sum_{k=0}^{\infty} \frac{(\Delta^k \cancel{x^n} f)}{k!}\Bigg|_0 x^{\underline{k}} = f(x) = (E^x f)(0) =$$

$$= (\Delta + 1)^x f(0) =$$

$$\underset{\text{Binom.}}{=} \left(\sum_{k=0}^{x} \binom{x}{k} \Delta^k f\right)(0)$$

$$= \sum_{k=0}^{x} \frac{x^{\underline{k}}}{k!} \Delta^k f \Bigg|_0$$

$$\Delta e^{x\lambda} = \lambda e^{\lambda x}$$

$$(E-1) f_\lambda(x) = \lambda f_\lambda(x)$$

$$f_\lambda(x+1) = (\lambda+1) f_\lambda(x)$$

$$f_\lambda(x) = (\lambda+1)^x f(0)$$

$$e^x = e^{\sum_{k=0}^{A} \frac{x}{A}} = \prod_{j=0}^{A} \underbrace{e^{\frac{x}{A}}}_{1+\frac{x}{A}+O\left(\left(\frac{x^2}{A}\right)\right)} = \left(1+\frac{x}{A}+O\left(\left(\frac{x}{A}\right)^2\right)\right)^A$$

$\rightarrow$ Taylor

$$\sum x^{\underline{n}} = \begin{cases} \frac{x^{\underline{n+1}}}{\underline{n+1}} & n \neq -1 \\ H(x) & n = -1 \end{cases}$$

$\rightarrow$ Harmonische Zahlen

$$P(\mathbb{Z}) \subset \mathbb{Z} \Leftarrow \sum_k \underbrace{\alpha_k}_{\in \mathbb{Z}} x^{\underline{k}}$$

p Polynom vom Grad n

$P = \sum_{k=0}^{n} \alpha_k \binom{x}{k}$ mit α_k ganzzahlig.

$$\frac{x^{\underline{k}}}{k!}$$

$$\alpha_k = \frac{\Delta^k p(0)}{k!}$$

$$x^{\overline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \,, \quad x^{\underline{n}} = \sum_k \left[\begin{matrix} n \\ k \end{matrix} \right] \sum_j \left\{ \begin{matrix} k \\ j \end{matrix} \right\} x^{\underline{j}}$$

$$x^n = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} \qquad = \sum_j \left(\sum_k \left[\begin{matrix} n \\ k \end{matrix} \right] \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \right) x^{\underline{j}}$$