Symmetrische Funktionen

Def:

$x = (x_1, x_2, \dots)$

$k \in \mathbb{N}$

$f(x) = \sum_\alpha c_\alpha x^\alpha$, $\alpha = (\alpha_1, \alpha_2, \dots)$, $\alpha_i \in \mathbb{N}$

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots$, $\sum_i \alpha_i = k$

$c_\alpha \in R$ (Ring) , z.B. $\mathbb{Q}$

$f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$, σ Permutation

$\Lambda^k = \{$ homogen symm. Fkt. vom Grad $k\}$

$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \dots$

$k \in \mathbb{N}$, $\underbrace{\lambda = (\lambda_1, \lambda_2, \dots)}_{\text{Partition von } k}$, $\lambda_i \in \mathbb{N}$, $\sum_i \lambda_i = k$, $\lambda_1 \geq \lambda_2 \geq \dots$

Bsp: $Par(1) = \{1\}$

$Par(4) = \{4, 31, 22, 211, 1111\}$

$Par(5) = \{5, 41, 32, 3, 221, 2111, 11111\}$

Menge der Partitionen von 5

$p(n)$: Anzahl der Partitionen von n

$p(4) = 5$

$p(5) = 7$

$$\sum_{n=0}^\infty p(n) t^n = \prod_k \left(\frac{1}{1-t^k}\right) = \prod_k \left(\sum_{\ell=0}^\infty (t^k)^\ell\right) = (1 + t + t^2 + \dots) \cdot (1 + t^2 + t^4 + \dots) \cdot \dots$$

Koeff. von t^n: $p(n)$; $n = a_1 + 2 \cdot a_2 + 3a_3 + \dots$

$\underbrace{1 + \dots + 1}_{a_1\text{-mal}} + \underbrace{2 + \dots + 2}_{a_2\text{-mal}} + \dots$

Monomial symmtr. Fkt.

$\lambda = (\lambda_1, \lambda_2, \dots)$ $\sum_i \lambda_i = n$

$m_\lambda = \sum_\alpha x^\alpha$

Λ^n $\rightarrow \in$ verschiedenen Permutationen von λ $\alpha(\alpha_1, \alpha_2, \dots)$

$m_1 = \sum_i x_i$

$((1, 0, \dots) + (0, 1, 0, \dots))$

$m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_4 + \dots + x_2^2 x_3 + \dots$

$\{m_\lambda \mid \sum \lambda_i = n\}$ Basis von Λ^n

$f = \sum_\alpha c_\alpha x^\alpha$ symmetrisch vom Grad n

$= \sum_{\lambda \text{ (Partition von } n)} c_\lambda m_\lambda$

Elementarsymmetrische Funktionen

$e_n = m_{1^n} = m_{\underbrace{1,\dots,1}_{n\text{-mal}}} = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}$, $n \in \mathbb{N}$

$e_\lambda := e_{\lambda_1} \cdot e_{\lambda_2} \cdots$, λ Partition von n $(\lambda \vdash n)$

$\{e_\lambda \mid \lambda \vdash n\}$ Basis von Λ^n

$e_1, e_2, \dots$ algebraisch unabhängig

$\Lambda = \mathbb{Q}[e_1, e_2, \dots]$

Bsp. $(x+y)^2 = x^2 + 2xy + y^2$
$= e_1^2$
$x^2 + y^2 = (x+y)^2 - 2xy$
$= e_1^2 - 2e_2$
$= e_{11} - 2e_2$

Bsp: $e_1 = x_1 + x_2 + \dots + x_n + \dots$

$e_2 = x_1 x_2 + x_1 \cdot x_3 + \dots + x_2 \cdot x_3 + \dots$

$e_3 = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + \dots + x_2 \cdot x_3 \cdot x_4 + \dots$

$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t) = (1 + x_1 t) \cdots = 1 + \underbrace{(x_1 + x_2 + \dots)}_{= e_1} t + \underbrace{(x_1 \cdot x_2 + x_1 \cdot x_3 + \dots)}_{= e_2} t^2 + \dots$

Vollständig symmtr. Fkt.

$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}$

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$, $\lambda = (\lambda_1, \lambda_2, \dots)$

$h_1 = e_1$

$h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + \dots + x_2^2 + x_2 x_3 + \dots$

$\{h_\lambda \mid \lambda \vdash n\}$ Basis von Λ^n

$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t} = \prod_i (1 + x_i t + x_i^2 t^2 + \dots)$

$= 1 + \underbrace{(x_1 + \dots + x_n + \dots)}_{= h_1} t + \underbrace{(x_1^2 + x_1 x_2 + \dots)}_{h_2} t^2 + \dots$

Potenzsummen

$p_n := m_n = \sum_i x_i^n$, $n \geq 1$

$p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2}$

$\{p_\lambda \mid \lambda \vdash n\}$ Basis von Λ^n (nur in $\mathbb{Q}$, nicht mehr in $\mathbb{Z}$)

$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t) = \exp(\ln(\prod_i (1 + x_i t))) = \exp(\sum_i \ln(1 + x_i t))$

$= \exp(\sum_i (\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} x_i t^k)) = \exp(\sum_k \frac{(-1)^{k+1}}{k} (\sum_i x_i^k) t^k)$

$$= \exp\left(\sum_k \frac{(-1)^{k+1}}{k} p_k t^k\right)$$

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_i (1-x_i t)^{-1} = \exp\left(\ln\left(\prod_i \frac{1}{1-x_i t}\right)\right)$$

$$= \exp\left(- \sum_i \ln(1-x_i t)\right) = \exp\left(-\sum_i \sum_k \frac{(-1)^{k+1}}{k} (-x_i t)^k\right)$$

$$= \exp\left(\sum_i \sum_k \frac{1}{k} x_i^k t^k\right) = \exp\left(\sum_k \frac{1}{k} t^k \sum_i x_i^k\right) = \exp\left(\sum_k \frac{1}{k} t^k p_k\right)$$

$$P(t) = \sum_{k\geq 1} p_k t^k = \sum_{k\geq 1} \sum_{i\geq 1} (x_i t)^k = \sum_{i\geq 1}\left(\sum_{k=0} (x_i t)^k - 1\right) = \sum_{i\geq 1} \frac{1}{1-x_i t} - 1$$

$$= \sum_{i\geq 1} \frac{x_i t}{1-x_i t}$$

$$E'(t) = \left(\prod (1+x_i t)\right)' = E(t) \sum_{i=1} \frac{-x_i}{1+x_i t}$$

$$\cancel{t\cdot E'(-t) = t\cdot E(-t) \cdot \sum_{i\geq 1} \frac{-x_i}{1-x_i t}}$$

$$t\cdot E'(t) = -E(t) P(-t)$$

$$\sum_{k\geq 0} t^k k e_k = \sum_{k\geq 0}\left(\sum_{i\geq 1} (-1)^{i-1} e_{k-i} p_i\right) t^k$$

$$t\cdot E'(t) = -\sum_k t^k \underbrace{\sum_{i=1}^{\infty} (-1)^i e_{k-i} p_i}_{= -E(t)P(-t)}$$

$$\Rightarrow k \cdot e_k = \sum_{i\geq 1} (-1)^{i-1} e_{k-i} p_i$$

$$\underbrace{\sum t^k h_k}_{H'(t)\, t} = \underbrace{\sum_k \sum_{i=1}^{k} h_{k-1} p_i t^k}_{H(t)P(t)} \qquad H'(t)\cdot t = H(t) P(t)$$

A Matrix

Ann: A diagonalisierbar, also A $\underset{=:\Lambda}{\text{ähnlich zu}} \begin{pmatrix}\lambda_1 & & 0 \\ & & \\ 0 & & \lambda_k\end{pmatrix}$

$$A^k \hat{=} \begin{pmatrix}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_k^k\end{pmatrix}$$

$$\mathrm{spur}(A^k) = \lambda_1^k + \dots + \lambda_k^k = p_k(\lambda_1, \lambda_2, \dots)$$

$$\det(tE_n - A) = \prod_i (t-\lambda_i) = t^n + \sum_{k=0}^{n-1} x^k (-1)^k e_{n-k}(\lambda_1 \dots \lambda_n)$$